New Specification

Rewarding Learning

ADVANCED SUBSIDIARY (AS)
General Certificate of Education
2019

Further Mathematics

Assessment Unit AS 2
assessing
Applied Mathematics

[SFM21]

MONDAY 17 JUNE, MORNING

MARK SCHEME

GCE ADVANCED/ADVANCED SUBSIDIARY (AS) FURTHER MATHEMATICS

Introduction

The mark scheme normally provides the most popular solution to each question. Other solutions given by candidates are evaluated and credit given as appropriate; these alternative methods are not usually illustrated in the published mark scheme.

The marks awarded for each question are shown in the right-hand column and they are prefixed by the letters **M**, **W** and **MW** as appropriate. The key to the mark scheme is given below:

M indicates marks for correct method.

W indicates marks for working.

MW indicates marks for combined method and working.

The solution to a question gains marks for correct method and marks for an accurate working based on this method. Where the method is not correct no marks can be given.

A later part of a question may require a candidate to use an answer obtained from an earlier part of the same question. A candidate who gets the wrong answer to the earlier part and goes on to the later part is naturally unaware that the wrong data is being used and is actually undertaking the solution of a parallel problem from the point at which the error occurred. If such a candidate continues to apply correct method, then the candidate's individual working must be followed through from the error. If no further errors are made, then the candidate is penalised only for the initial error. Solutions containing two or more working or transcription errors are treated in the same way. This process is usually referred to as "follow-through marking" and allows a candidate to gain credit for that part of a solution which follows a working or transcription error.

Positive marking:

It is our intention to reward candidates for any demonstration of relevant knowledge, skills or understanding. For this reason we adopt a policy of **following through** their answers, that is, having penalised a candidate for an error, we mark the succeeding parts of the question using the candidate's value or answers and award marks accordingly.

Some common examples of this occur in the following cases:

(a) a numerical error in one entry in a table of values might lead to several answers being incorrect, but these might not be essentially separate errors;

(b) readings taken from candidates' inaccurate graphs may not agree with the answers expected but might be consistent with the graphs drawn.

When the candidate misreads a question in such a way as to make the question easier only a proportion of the marks will be available (based on the professional judgement of the examining team).

SECTION A: Mechanics 1

AVAILABLE MARKS

1 (a) $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$ MW1

Work done = $\mathbf{F} \bullet \overrightarrow{AB}$

$= \begin{pmatrix} 7 \\ -2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 2 \\ -5 \end{pmatrix}$ M1 W1

$= 21 - 4 - 5 = 12\,\text{J}$ W1

(b) (i) Work done $= \int_{1.5}^{d} F \, dx$

$= \int_{1.5}^{d} \left(8x - \frac{18}{x^3}\right) dx$ M1 W1

$= \left[4x^2 + 9x^{-2}\right]_{1.5}^{d}$ MW1

$= 4d^2 + \frac{9}{d^2} - 13$ W1

(ii) $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \int_{1.5}^{d} F \, dx$

$\frac{1}{2}(3 \times 16) - 0 = 4d^2 + \frac{9}{d^2} - 13$ M1 W1

$0 = 4d^4 - 37d^2 + 9$ MW1

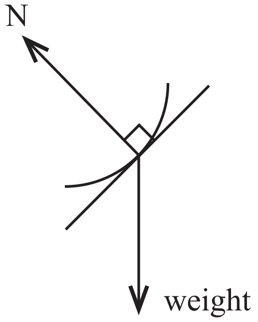
$0 = (4d^2 - 1)(d^2 - 9)$

$d = \pm\frac{1}{2}, \pm 3$

$d > 1.5$ so $d = 3$ W1 12

2 **(i)** Normal ⊥ tangent
radius ⊥ tangent
∴ Normal acts along radius through O — MW1

(ii)



MW2

(iii) Let mass of mouse be m

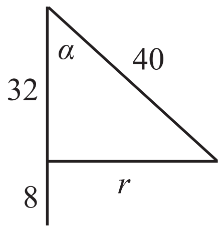
↕ $N \cos \alpha = mg$ — M1 W1

↔ $N \sin \alpha = m\frac{v^2}{r}$ — M2 W1

dividing $\tan \alpha = \frac{v^2}{rg}$ — M1

$v^2 = rg \tan \alpha$ — W1

(iv)



$r = \sqrt{40^2 - 32^2} = 24\,\text{cm}$ — MW1

$\tan \alpha = \frac{24}{32} = \frac{3}{4}$

$v^2 = 0.24 \times 9.8 \times 0.75 = 1.764$

$v = 1.33\,\text{m s}^{-1}$ — MW1 — 12

3 Cross section of pipe $= 0.14^2\pi\ \text{m}^2$

Volume of water per sec $= 0.14^2 \times 5 \times \pi\ \text{m}^3\,\text{s}^{-1}$ — M1 W1

Mass of water per sec at 25 m $= 0.14^2 \times 5000\pi\ \text{kg s}^{-1}$

$= 98\pi\ \text{kg s}^{-1}$ — MW1

PE created at 25 m per sec $= 98\pi \times 9.8 \times 25$ — M1 W1

KE at 25 m per sec $= \frac{1}{2}(98\pi)\,25$ — M1 W1

Effective power of the pump

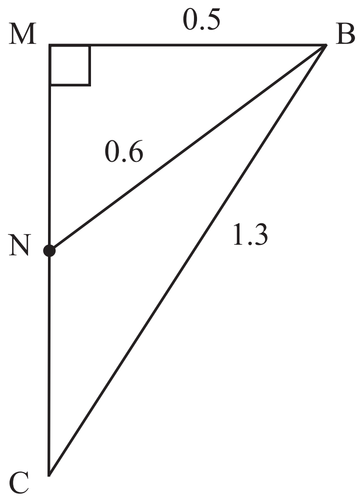
$= 98\pi \times 25 \times (9.8 + 0.5)$ — M1

$= 25\,235\pi$ — W1 — 9

$= 79\,300\ \text{W or } 79.3\ \text{kW}$

AVAILABLE MARKS
12
9

4



AVAILABLE MARKS

(i) At C:

Kinetic Energy = $KE_C = 0$

Gravitational Potential Energy = $GPE_C = 0$ — M1

Elastic Potential Energy = $2 \times \dfrac{400 \times 0.7^2}{2 \times 0.6}$ — W1 MW1

$E_C = KE_C + GPE_C + EPE_C = 326\frac{2}{3}$ J — W1

(ii) $MN = \sqrt{0.6^2 - 0.5^2} = \sqrt{0.11} = 0.33166$ — M1

$MC = \sqrt{1.3^2 - 0.5^2} = 1.2$

$NC = 1.2 - 0.33166 = 0.86834$ — W1

At N: $KE_N = \frac{1}{2} \times 10 \times v^2$ — MW1

$EPE_N = 0$ — M1

$GPE_N = 10 \times 9.8 \times 0.86834 = 85.097$ — M1 W1

By Conservation of Energy

$E_C = E_N$ — M1

so $5v^2 + 85.097 = 326\frac{2}{3}$ — MW1

$v = 6.95\,\text{m}\,\text{s}^{-1}$ — W1

(iii) Free motion of 9 kg under gravity with initial upward velocity of v from N — M1

$\frac{1}{2} 9v^2 + 9g\,[NC] = 9gh$ — MW2

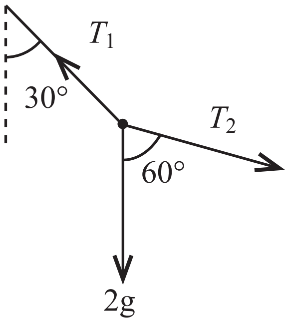
$h = NC + \dfrac{v^2}{2g}$

$= 3.33$m above GPE base — W1 — 17

Section A — **50**

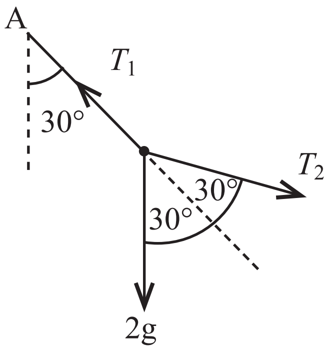
SECTION B: Mechanics 2

		Marks	AVAILABLE MARKS
1	Resolving $\leftrightarrow$ $T_1 \sin 30 = T_2 \sin 60$	M1 W1	
	$T_1 = \sqrt{3}\, T_2 \quad (1)$		



		Marks	AVAILABLE MARKS
$\updownarrow$	$2g + T_2 \cos 60 = T_1 \cos 30$	M1 W1	
	$2g + \frac{1}{2}\, T_2 = \frac{\sqrt{3}}{2}\, T_1$		
Sub (1)	$2g + \frac{1}{2}\, T_2 = \frac{3}{2}\, T_2$	M1	
	$T_2 = 2g$ N		
Using (1)	$T_1 = 2\sqrt{3}g$ N	W1	
Hooke's Law	$\dfrac{0.1\lambda}{0.4} = T_1$	M1	
	$\therefore\ \lambda = 8\sqrt{3}\,g$N	W1	8

Alternative Solution



	Marks
Consider along AP	M2
Symmetry $\Rightarrow T_2 = 2g$	M1 W1
Along AP $T_1 = 2g \times 2 \cos 30$	
$= 2\sqrt{3}\,g$	M1 W1
Hooke's Law $\dfrac{\lambda}{4} = 2\sqrt{3}\,g$	M1
$\lambda = 8\sqrt{3}\,g$ N	W1

2 **(i)** Consider $F = \eta \dfrac{As}{y}$

take dimensions — M1

$$[\eta] = \frac{[F][y]}{[A][s]}$$

$[F] = \text{ML T}^{-2}$ $\quad [y] = \text{L}$ — MW1

$[A] = \text{L}^2$ $\quad [s] = \text{LT}^{-1}$ — MW1

$$[\eta] = \frac{\text{ML T}^{-2}\text{ L}}{\text{L}^2\text{ L T}^{-1}} = \text{ML}^{-1}\text{ T}^{-1}$$ — W1

(ii) $Q = k\, R^a\, L^b\, P^c\, \eta^d$

$[Q] = \text{L}^3\ \text{T}^{-1}$ — MW1

$[P] = \dfrac{\text{ML T}^{-2}}{\text{L}^2} = \text{ML}^{-1}\text{ T}^{-2}$ — MW1

$[R] = [L] = \text{L}$ and $[\eta] = \text{ML}^{-1}\text{ T}^{-1}$

$\text{L}^3\,\text{T}^{-1} = \text{L}^a\,\text{L}^b\,(\text{ML}^{-1}\text{ T}^{-2})^c(\text{ML}^{-1}\text{ T}^{-1})^d$ — M1

Compare indices M: $0 = c + d$ — M1

L: $3 = a + b - c - d$

T: $-1 = -2c - d$ — W1

$\therefore\ a + b = 3 \quad c = 1 \quad d = -1$ — W1 W1 — 11

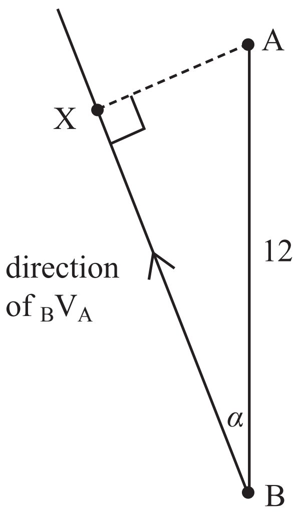
3 Universal Law of Gravitation

		AVAILABLE MARKS
Force = $\dfrac{GM_PM_S}{R^2}$	M1 W1	
radial acceleration = $\omega^2 R$	MW1	
Newton's second law for planet:		
$M_P\omega^2 R = \dfrac{GM_PM_S}{R^2}$	M1 W1	
$\omega^2 = \dfrac{GM_S}{R^3}$	W1	
$T = \dfrac{2\pi}{\omega}$	M1	
$T^2 = \dfrac{4\pi^2}{GM_S} R^3$		
constant of proportionality = $\dfrac{4\pi^2}{GM_S}$	MW1	8

4 **(i)** $\mathbf{v}_A = \begin{pmatrix} 18 \\ -5 \end{pmatrix}$ $\mathbf{v}_B = \begin{pmatrix} 12 \\ 16 \end{pmatrix}$

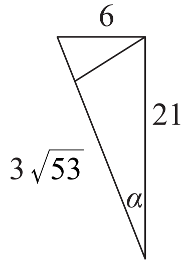
		AVAILABLE MARKS
Consider motion relative to A	M1	
${}_B\mathbf{v}_A = \mathbf{v}_B - \mathbf{v}_A = \begin{pmatrix} 12 \\ 16 \end{pmatrix} - \begin{pmatrix} 18 \\ -5 \end{pmatrix} = \begin{pmatrix} -6 \\ 21 \end{pmatrix}$	W1 W1	

Displacement Diagram:



direction of ${}_B\mathbf{v}_A$ MW1

perp. at X M1



Closest approach

$AX = 12 \sin \alpha = 12 \times \frac{2}{\sqrt{53}} = 3.29665\,\text{km}$ M1 W1

$= 3.30\,\text{km}$

(ii) Dist $BX = 12 \cos \alpha = \frac{84}{\sqrt{53}}$ MW1

Relative speed $= |{}_B\mathbf{v}_A| = 3\sqrt{53}$ MW1

Time to X $= \frac{12 \cos \alpha}{|{}_B\mathbf{v}_A|}$ M1

$= 12 \times \frac{7}{\sqrt{53}} \times \frac{1}{3\sqrt{53}}$

$= 0.5283\text{ hr} \rightarrow 1632\text{ hours}$ W1 11

				AVAILABLE MARKS
5	**(i)**	At P, by conservation of energy	M1	
		$\frac{1}{2}mu^2 = \frac{1}{2}mv^2 + mgr\cos\theta$	M1 W2	
		$v^2 = u^2 - 2gr\cos\theta$		
		$v = \sqrt{u^2 - 2gr\cos\theta}$	W1	
	(ii)	Resolve ‖ radius OP	M1	
		$\frac{mv^2}{r} = mg\cos\theta - N$	M1 MW1	
		$N = mg\cos\theta - \frac{mv^2}{r}$		
		$= mg\cos\theta - \frac{mu^2}{r} + 2mg\cos\theta$	MW1	
		$= 3mg\cos\theta - \frac{mu^2}{r}$	W1	
	(iii)	$N = 0 \Rightarrow \cos\theta = \frac{u^2}{3gr}$	M1	
		$= \frac{21^2}{3 \times 9.8 \times 10\sqrt{3}}$		
		$\Rightarrow \theta = 30°$	W1	12
			Section B	**50**

SECTION C: Statistics

AVAILABLE MARKS

1 **(i)** t = time which is the explanatory variable because velocity depends on it. M1
v = speed which is the response variable because it responds to changes to the explanatory variable. M1

(ii) From calculator

$\Sigma t = 440$ $\quad \Sigma v = 595$
$\Sigma t^2 = 28\,400$ $\quad \Sigma v^2 = 47\,573$ $\quad \Sigma tv = 36\,430$ M1 W1
$b = 0.882$ $\quad \bar{x} = 55,\ \bar{y} = 74.375$ M1 W1
$a = \bar{y} - b\bar{x}$
$a = 25.86 = 25.9$ (3 s.f.) W1

$v = 25.9 + 0.882t$ MW1

Alternative solution: Using formula

$\Sigma t = 440$ $\quad \Sigma v = 595$
$\Sigma t^2 = 28\,400$ $\quad \Sigma v^2 = 47\,573$ $\quad \Sigma tv = 36\,430$ M1 W1

$$b = \frac{\Sigma tv - \frac{\Sigma t\, \Sigma v}{n}}{\Sigma t^2 - \frac{(\Sigma t)^2}{n}}$$

$$b = \frac{36430 - \frac{440 \times 595}{8}}{28\,400 - \frac{(440)^2}{8}}$$ M1

$b = 0.882$ $\quad \bar{x} = 55,\ \bar{y} = 74.375$ W1
$a = \bar{y} - b\bar{x}$
$a = 25.86 = 25.9$ (3 s.f.) W1

$v = 25.9 + 0.882t$ MW1

(iii) $t = 65$, $v = 83.19$ M1 W1

(iv) The scores for time have a maximum value of $t = 90$ seconds therefore it would be unsafe to extrapolate as far as 120 sec. M1 11

				AVAILABLE MARKS
2	**(i)**	$\int_0^2 kx\,(4-x)\,dx = 1$	M1	
		$\int_0^2 4kx - kx^2\,dx = 1$	M1	
		$\left[\frac{4kx^2}{2} - \frac{kx^3}{3}\right]_0^2 = 1$	W1	
		$\left[\frac{12k(2)^2 - 2k(2)^3}{6}\right] - 0 = 1$		
		$48k - 16k = 6$		
		$32k = 6$		
		$k = \frac{6}{32} = \frac{3}{16}$	W1	
	(ii)	$E(X) = \int_0^2 kx^2\,(4-x)\,dx = \frac{3}{16}\left[\frac{4x^3}{3} - \frac{x^4}{4}\right]_0^2$	M1	
		$E(X) = 1.25$	W1	
		$Var(X) = E(X^2) - E(X)^2$		
		$E(X^2) = \int_0^2 kx^3\,(4-x)\,dx$	MW1	
		$= \frac{3}{16}\left[\frac{4x^4}{4} - \frac{x^5}{5}\right]_0^2$		
		$= \frac{3}{16}\left[\frac{4(16)}{4} - \frac{32}{5}\right] = 1.8$		
		$Var(X) = 1.8 - 1.25^2$	M1	
		$= 0.2375$	W1	
		$= 0.238$ (3 s.f.)		
	(iii)	Modal value when graph at a maximum		
		$f(x) = \frac{3}{16}x(4-x)$		
		$f(x) = \frac{3}{4}x - \frac{3}{16}x^2$		
		$f'(x) = \frac{3}{4} - \frac{3}{8}x$	M1	
		$\frac{3}{4} - \frac{3}{8}x = 0$	MW1	
		$\frac{3}{4} = \frac{3}{8}x$		
		$x = 2$	W1	
		$f''(x) = -\frac{3}{8}$ which is negative, therefore maximum (or graph sketch)	MW1	13

AVAILABLE MARKS

3 **(i)** $X \sim Po(1.75)$

$P(X = 3) = \frac{e^{-1.75} 1.75^3}{3!} = 0.15522$

$= 0.155$ (3 s.f.) M1 W1

(ii) $0.15522 \times 0.15522 = 0.024093$

$= 0.0241$ (3 s.f.) M1 W1

(iii) $\lambda = 1.75 \times 2 = 3.5$ per 2 weeks MW1

$P(X \geqslant 4) = 1 - (X = 0, X = 1, X = 2, X = 3)$ M1 W1

$= 1 - \frac{e^{-3.5} 3.5^0}{0!} - \frac{e^{-3.5} 3.5^1}{1!} - \frac{e^{-3.5} 3.5^2}{2!} - \frac{e^{-3.5} 3.5^3}{3!}$ W1

$= 1 - 0.536633$

$= 0.463$ (3 s.f.) W1

Alternative solution: from tables

$\lambda = 1.75 \times 2 = 3.5$ per 2 weeks MW1

$P(X \geqslant 4) = 1 - (X = 0, X = 1, X = 2, X = 3)$ M1 W1

$= 1 -$ (probabilities from tables) W1

$= 0.463$ (3 s.f.) W1 9

4 **(i)** The Human Resources department should use stratified sampling, placing each of the employees into well-defined strata based on job role. For example, factory staff, maintenance staff, office staff, managerial staff, part-time staff, full-time staff, gender, etc. MW1

A random selection of each group of workers is sampled for their opinions on workforce conditions. MW1

An appropriate proportion of workers from each strata should be collected. MW1

(ii) Advantage: This should be a relatively simple technique for identifying workforce to be sampled. MW1

Minimises sample selection bias, ensuring a greater level of representation (certain segments of the population are not underestimated or overestimated).

Disadvantage: Some people may be classified into two categories.

Could be time-consuming. MW1

Alternatively, if choose wrong type of sampling.

Random sampling: Advantage: there is no bias in the population, each staff member has an equal chance of being selected. Quick to obtain the sample.

Disadvantage: Sample may not be representative of the whole population. MW2 5

			AVAILABLE MARKS
5	**(i)** $^{10}C_4 = 210$	M1 W1	
	(ii) To include the fastest male and fastest female, No. of ways of selecting 2 from remaining 8 is $^{8}C_2 = 28$	MW2	
	(iii) prob (both fastest male and female selected) $= \frac{28}{210} = \frac{2}{15} = 0.133$ (3 s.f.)	M1 W1	
	(iv) No. of ways to choose 2 men is $^{4}C_2 = 6$ No. of ways to choose 2 females is $^{6}C_2 = 15$ Total number of ways $= 6 \times 15 = 90$	MW1 MW1 W1	
	(v) Fastest male and fastest female selected: 1 man from 3 is $^{3}C_1 = 3$ 1 female from 5 is $^{5}C_1 = 5$ Total number of ways $3 \times 5 = 15$ Probability (fastest male and fastest female) $= \frac{15}{90} = \frac{1}{6}$	 MW1 MW1 W1	12
		Section C	**50**

SECTION D: Discrete and Decision Mathematics

				AVAILABLE MARKS
1	**(a)**	**(i)**	A Hamiltonian cycle is: a closed tour of vertices M1 visiting every vertex once only M1	
		(ii)	a – c – d – f – e – i – h – j – g – b – a start and finish on correct letter MW1 a Hamiltonian cycle MW1	
	(b)	**(i)**	An Eulerian circuit is: a closed tour of edges M1 visiting every edge once only M1	
		(ii)	a – b – c – d – e – f – g – a – d – g – c – f – b – e – a MW2	8

2 **(i)** Substitute $P_n = kx^n$

obtaining $kx^{n+2} = 2kx^{n+1} + kx^n$ — M1

rearrange to $x^2 - 2x - 1 = 0$ — W1

(ii) $x^2 - 2x - 1 = 0$

$x = (2 \pm \sqrt{4+4})/2 = 1 \pm \sqrt{2}$ — MW1

Then $P_n = A(1 + \sqrt{2})^n + B(1 - \sqrt{2})^n$ — M1

$P_0 = 0 = A + B$

and $P_1 = A(1 + \sqrt{2}) + B(1 - \sqrt{2}) = 1$ — M1 W1

Solving $A + B = 0$ — M1

$A - B = \frac{1}{\sqrt{2}}$

we obtain $A = \frac{1}{2\sqrt{2}}$ and $B = \frac{-1}{2\sqrt{2}}$ — W1

Thus $P_n = \frac{1}{2\sqrt{2}}(1 + \sqrt{2})^n - \frac{1}{2\sqrt{2}}(1 - \sqrt{2})^n$

$= \frac{1}{2\sqrt{2}}\left\{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n\right\}$ — MW1 — 9

3

p	q	r	$\sim q$	r or $\sim q$	p and (r or $\sim q$)	p and r	p and $\sim q$	(p and r) or (p and $\sim q$)
T	T	T	F	T	T	T	F	T
T	T	F	F	F	F	F	F	F
T	F	T	T	T	T	T	T	T
T	F	F	T	T	T	F	T	T
F	T	T	F	T	F	F	F	F
F	T	F	F	F	F	F	F	F
F	F	T	T	T	F	F	F	F
F	F	F	T	T	F	F	F	F

AVAILABLE MARKS

Set up 1st 3 columns — M1 W1
Columns 4, 5, 6, 7, 8 — 5 × W1
Stating column 6 = column 9 — MW1 — 8

4 **(a)** Edges in $K_6 = 6 \times 5 \div 2 = 15$ — MW1
Edges in spanning tree = 5 (to connect 6 vertices) — MW1
Need to remove 15 – 5 = 10 edges — MW1

(b) **(i)** Start with vertex A

+ edge AE — correct choice of 1st edge — M1

+ edge ED — MW1

+ edge DB — MW1

+ edge DC — correct choice — M1

+ edge CF — W1

(ii)
84
85
119
148
147
583 km — W1 — 9

				AVAILABLE MARKS
5	**(i)** $B \otimes (C \otimes E) = B \otimes G$			
	$(B \otimes C) \otimes E = B \otimes G$	using associativity	M1	
	$D \otimes E = H$	from table	W1	

(ii)

⊗	A	B	C	D	E	F	G	H
A	A	B	C	D	E	F	G	H
B	B	A	D	C	F	E	H	G
C	C	D	B	A	G	H	F	E
D	D	C	A	B	H	G	E	F
E	E	F	H	G	B	A	C	D
F	F	E	G	H	A	B	D	C
G	G	H	E	F	D	C	B	A
H	H	G	F	E	C	D	A	B

Using Latin Square Property — M1
Six entries correct — W1
All entries correct — W1

(iii) Order of A = 1 — M1
Order of B = 2 — W1
Order of C, D, E, F, G, H = 4 — W2

(iv) Q has no order 8 elements, while R has 4 — M1
Therefore they are not isomorphic. — W1

(v) Subgroup of (Q, ⊗)

Element	A	C	B	D
Period	1	4	2	4

Identifies correct subgroup — M1
Orders correct — W1

Subgroup of (R, ✶)

Element	e	g^2	g^4	g^6
Period	1	4	2	4

Identifies correct subgroup with orders correct — MW1
Isomorphism: $A \to e$ $B \to g^4$ $C \to g^2$ $D \to g^6$ — M1 W1 — 16

Alternative solution
$A \to e$ $B \to g^4$ $C \to g^6$ $D \to g^2$

Section D	**50**
Total	**100**